HomeWork 2 : Representation of the spin

Convention

If nothing is precised, I will used a, $a^\dagger$ for bosonic annihilation, creation operators, such that they fulfil the commutation relation $[a_\alpha, a^\dagger_\beta] = \delta_{\alpha\beta}$ (α, β being any quantum label of my states). And c, $c^\dagger$ will in principle refer to fermionic annihilation, creation operators, governed by an anticommutation relation $\{c_\alpha, c^\dagger_\beta\} = \delta_{\alpha\beta}$.

1 Natural representation of the spin

Making use of the Pauli matrix identity $\sigma_{\alpha\beta} \cdot \sigma_{\gamma\delta} = 2\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}$ (where "." denotes the scalar or dot product), prove that

$$\hat{S}_m \cdot \hat{S}_n = -\frac{1}{2}\sum_{\alpha\beta} c^\dagger_{m\alpha} c^\dagger_{n\beta} c_{m\beta} c_{n\alpha} - \frac{1}{4}\hat{n}_m \hat{n}_n$$

where $\hat{S}_n = (1/2)\sum_{\alpha\beta} c^\dagger_{m\alpha}\sigma_{\alpha\beta}c_{m\beta}$ denotes the spin operator, and $\hat{n}_m = \sum_\alpha c^\dagger_{m\alpha}c_{m\alpha}$ represents the total number operator on site m. (NB : here we assume that lattice sites m and n are distinct).

2 The Holstein-Primakoff transformation

In several problems of magnetism where the spin S is large, it exists a useful representation of the spin known as the Holstein-Primakoff transformation. Within this representation, the spin raising and lowering operators are specified in terms of boson creation and annihilation operators $a^\dagger$ and a. Starting from the definition

$$\hat{S}^- = (2S)^{1/2} a^\dagger \left(1 - \frac{a^\dagger a}{2S}\right)^{1/2}$$

and

$$\hat{S}^z = S - a^\dagger a$$

confirm the validity of the Holstein-Primakoff transformation by explicitly checking the commutation relations of the spin raising and lowering operators ($[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$).

3 The Jordan-Wigner transformation

In exercice 2, we have seen a way to express the quantum spin algebra in terms of boson operators. In this exercice, we show that a representation for spin 1/2 can be obtained in terms of fermion operators. Specifically, let us formally represent an up spin as a particle and a down spin as the vaccum, namely :

$$\begin{aligned} |\uparrow\rangle &\equiv |1\rangle = f^\dagger|0\rangle \\ |\downarrow\rangle &\equiv |0\rangle = f|1\rangle \end{aligned}$$

In this representation the spin raising and lowering operators are expressed in the form $\hat{S}^+ = f^\dagger$ and $\hat{S}^- = f$, while $\hat{S}^z = f^\dagger f - 1/2$

3.1 With this definition, confirm that the spins obey the $SU(2)$ algebra $[\hat{S}^+, \hat{S}^-] = 2\hat{S}^z$.

However, there is a problem : spins on different sites commute while fermion operators anticommute, e.g.

$$\hat{S}_i^+ \hat{S}_j^+ = \hat{S}_j^+ \hat{S}_i^+, \quad \text{but } f_i^\dagger f_j^\dagger = -f_j^\dagger f_i^\dagger.$$

To obtain a faithful spin representation, it is necessary to cancel this unwanted sign. Although a general procedure is hard to formulate, in one dimension this can be achieved by a non-linear transformation :

$$\hat{S}_l^+ = f_l^\dagger e^{i\pi \sum_{j<l} \hat{n}_j}, \quad \hat{S}_l^- = e^{-i\pi \sum_{j<l} \hat{n}_j} f_l, \quad \hat{S}_l^z = f_l^\dagger f_l - 1/2.$$

This looks complicated but it's not. Have in mind this picture : in one dimension, the particles can be ordered on a line. By counting the number of particles "to the left" we can assign an overall phase of +1 or -1 to a given configuration and thereby transmute the particles into fermions. (In other words, the exchange of two fermions induces a sign change which is compensated by the factor arising from the phase - the so-called "*Jordan-Wigner string*".)

3.2 Using the Jordan-Wigner representation, show that

$$\hat{S}_m^+ \hat{S}_{m+1}^- = f_m^\dagger f_{m+1}.$$

3.3 Application : the Heisenberg quantum chain

For the spin 1/2 anisotropic quantum Heisenberg spin chain, the Hamiltonian is of the form :

$$H = -\sum_n \left[J_z \hat{S}_n^z \hat{S}_{n+1}^z + \frac{J_\perp}{2} (\hat{S}_n^+ \hat{S}_{n+1}^- + \hat{S}_n^- \hat{S}_{n+1}^+) \right]$$

Turning to the Jordan-Wigner representation, show that the Hamiltonian can be cast in the form :

$$H = -\sum_n \left[\frac{J_\perp}{2} (f_n^\dagger f_{n+1} + f_{n+1}^\dagger f_n) + J_z (\frac{1}{4} - f_n^\dagger f_n + f_n^\dagger f_n f_{n+1}^\dagger f_{n+1}) \right]$$

3.4 The mapping above shows that the one-dimensional quantum spin 1/2 XY model (i.e. $J_z = 0$) can be diagonalized as a non-interacting theory of spinless fermions. In this case, establish the dispersion relation (relation energy-momentum).